



WORKING PAPER No. 5/2009

**Risk-neutral Option Pricing and Information
Embedded in CEZ Warrant Prices: Implied
Volatility Smile, Implied Density Function**

Martin Cícha

12 2009



Řada studií Working Papers Centra výzkumu konkurenční schopnosti české ekonomiky je vydávána s podporou projektu MŠMT výzkumná centra 1M0524.

ISSN 1801-4496



RISK-NEUTRAL OPTION PRICING AND INFORMATION EMBEDDED IN CEZ WARRANT PRICES: IMPLIED VOLATILITY SMILE, IMPLIED DENSITY FUNCTION

Abstract.

Risk-neutral approach is one of two main approaches to derivatives pricing. This paper introduces a derivatives pricing tool derived from a binomial tree of underlying asset price under the assumption of constant volatility. This article focuses on stocks, but the proposed approach can be generalized and applied to other underlying assets as well. The assumption of constant volatility leads to a lognormal distribution of underlying asset price. However, the volatility observed in the market is often a function of option strike and time to maturity rather than a constant. Hence, using lognormal distribution, in pricing formulas, leads to incorrect prices. In this paper provide evidence of the nonlognormality of CEZ stock using American-style call warrants. As warrants are American-style, we have to take into account the possibility of early exercise. I construct a volatility smile time series for CEZ warrants. Further, I derive implied risk-neutral density function of CEZ stock price from observed warrants market prices. The resulting implied density function can be used for correct derivatives pricing. Implied density function also provides us with a useful way of analyzing market reactions and expectations of future stock price to company earnings releases, changes in official interest rate, government bond auctions, etc. I provide some expectation of future distribution of CEZ stock price using higher moments of implied density function.

Recenzovala: Ing. Irena Jindřichovská, CSc.



1. INTRODUCTION

Optional contracts are a large group of financial instruments and unique approach to their correct pricing has not been introduced yet. The replication strategy and risk-neutral approach are two main approaches that lead to identical results. Under the assumption of constant volatility we derive lognormal distribution of an underlying asset. Using lognormal distribution in pricing formulas we get the theoretical price of the option. It is often the case that traders use volatility as a function of option strike price and option time to maturity rather than as a constant value, which leads to volatility surface. Lognormal distribution of underlying asset can not be used for pricing any more. It is possible to use the information that is embedded in options prices to derive implied distribution of underlying asset. These density functions estimated from cross-section of observed option prices are gaining increasing attention. They are used for pricing complex derivatives and derivatives not traded in the market. Using implied distribution of underlying asset we can also find the mispriced derivatives and earn excess returns above the risk-free rate of interest by buying options that are undervalued by the market and selling options that are overvalued by the market. A number of authors have used implied density functions as indicator of market sentiment to examine whether options markets anticipated major economic events. Central banks, in particular, have been interested in using implied density functions to assess market participants expectations of future change in interest rates, stock prices and exchange rates. Many research papers on this may be found issue in research bulletins of central banks. See for example, FED and Bank of England (Bliss and Panigirtzoglou, 2002), Bank of England (Bahra, 1997), ECB (Schneider and Glatzer, 2003). The method for estimating implied density function fall into five groups: stochastic process methods, implied binomial trees, finite-difference methods, implied volatility smoothing methods and density function approximating methods. Stochastic process methods begin by assuming a model for stochastic process driving the prices of the underlying asset. This approach can be used in the absence of options prices. For instance, Hoerdahl applied the Longstaff-Swchartz model to Swedish interest rates in (Hoerdahl, 2000). The implied binomial tree method was developed by (Rubinstein, 1994). The smoothed implied volatility method was originally developed in (Shimko, 1993). This is an approximating function method to the volatility smile rather than to the density function. Approximating function methods are based on minimizing the difference between observed options prices and fitted prices produced by particular functional form, chosen to allow for a variety of possible shapes. For instance, Melick and Thomas used mixtures of lognormals in (Melick and Thomas, 1994), Mandan and Milne used Hermite



polynomials in (Mandan and Milne, 1994) and alternatively Ait-Sahalia used non-parametric kernel estimator in (Ait-Sahalia and Lo, 1998).

The paper is organized as follows. Section 1 introduces binomial tree model in the real world and shows the transition from the real to the risk-neutral world. Further, Section 1 discusses real and risk-neutral distribution of the stock price. Section 2 introduces the risk-neutral option pricing approach and applies it to the real market options. Section 3 discusses volatility in constant volatility assumption. Section 3 also shows that warrants on CEZ stock do not have constant volatility, hence CEZ stock price is not lognormally distributed at any future time. Furthermore, Section 3 introduces deriving volatility smile and volatility term structure from American-style call warrants on CEZ. Section 4 discusses two-lognormal mixture implied probability distribution of the CEZ stock price, shows an estimation of parameters and deals with the computing of higher moments. Using these statistics, we can make an assumption about future CEZ stock price. Section 5 is the conclusion.

2. THE BINOMIAL TREE MODEL AND EXPECTATION APPROACH

A derivative is the function of underlying asset, say stock, at a specific time in the future. Let us assume that the price of the stock follows the process of binomial tree. Due to this only two things can happen to the stock at each time t : an 'up' move or 'down' move. We start at the time $t = 0$ and continue onwards to the time $t = T$ using time step δt . We will assign probabilities to these moves: probability p to move up, and thus $1 - p$ to move down. We also need a cash bond B to represent the time-value of money. There will be some continuously compounded interest rate r that will hold for the period δt and will cause that 1 \$ today will be worth $e^{r\delta t}$ \$ a tick later.

We can use two approaches to the derivative pricing at this moment. The first one is the so-called replication strategy. The replication strategy is based on the fact that any derivative of a stock, derivative pay-off particularly, can be constructed in advance from an appropriate portfolio of bond and stock. The value of the derivative would exactly cancel the value of the portfolio whatever the stock price was at the end of the tick-period, see (Joshi, 2005) for details. Therefore, if one knows the value of this portfolio, one also knows the value of the derivative. The other derivatives pricing approach is based on expectation and change of the measure. We can imagine the value of the derivative as an expectation of discounted claim under martingale measure. I will deal only with the expectation approach in this paper.



Let us assume that the stock price process $\{S_t, t \geq 0\}$ has a constant growth rate μ and constant noise σ . The stock jumps from the value S_t to the values at time $t + \delta t$. The value $S_{t+\delta t}$ after up jump determines (1). The value $S_{t+\delta t}$ after down jump determines (2).

$$S_{t+\delta t}^{up} = S_t \exp(\mu \delta t + \sigma \sqrt{\delta t}) \quad (1)$$

$$S_{t+\delta t}^{down} = S_t \exp(\mu \delta t - \sigma \sqrt{\delta t}) \quad (2)$$

All jumps are equally likely in the real world, i.e. under a Π measure. The real world measure Π under which S_t occurs in the real world is irrelevant for correct pricing (Baxter, 1999). We have to find the measure Θ under which the process of discounted stock $\{B_t^{-1} S_t, t \geq 0\}$ is a martingale. Let us define the variable

$$q_t = \frac{S_t \exp(r \delta t - S_{t+\delta t}^{down})}{S_{t+\delta t}^{up} - S_{t+\delta t}^{down}} \quad (3)$$

If the condition of rational market is fulfilled, following relation holds $S_{t+\delta t}^{down} < S_t e^{r \delta t} < S_{t+\delta t}^{up}$ which force q_t into $(0,1)$. We demand the same constraint for probability. It can be shown that q_t is the martingale measure probability (Baxter, 1999). We can calculate q_t approximately equal to

$$q_t = \frac{1}{2} \left(1 - \sqrt{\delta t} \left(\frac{\mu + \frac{1}{2} \sigma^2 - r}{\sigma} \right) \right) \quad (4)$$

Consequently, q_t depends no longer on time t but only on time step δt . Hence, the whole tree has only one unique probability q . The binomial tree model is parameterized by the time step δt . As that quantity gets smaller, the model should ever more closely approximate the real world. For a fixed time t , we get stock value under Π .



$$S_t = S_0 e^{\left(\mu + \sigma \sqrt{t} \left(\frac{2X_n - n}{\sqrt{n}} \right) \right)}, \quad (5)$$

where n is the number of time steps till time t and X_n is the total number of the n separate jumps which were up-jumps. X_n is binomially distributed random variable with mean $\frac{n}{2}$ and variance $\frac{n}{4}$,

so that $\frac{2X_n - n}{\sqrt{n}}$ has zero mean and variance equals 1. Applying the central limit theorem, this distribution converges to the standard normal distribution. As δt gets smaller, the distribution of S_t becomes log-normal with mean $e^{1/2\sigma^2 t} S_0 e^{\mu t}$ and variance $S_0^2 \left(e^{2t(\mu + \sigma^2)} - e^{t(\sigma^2 + 2\mu)} \right)$.

Distribution of S_t is useless for correct pricing under real world Π measure and thus we apply the same procedure under martingale measure Θ . Using up-move probability, X_n is still binomially distributed with mean nq and variance $nq(1-q)$. Hence, $\frac{2X_n - n}{\sqrt{n}}$ has

mean $-\sqrt{t} \frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma}$ and variance asymptotically approaching one.

As δt gets smaller, the central limit theorem tells us that this formula converges to a normal variable with the same mean and variance exactly one. Then transforming this normal random variable we get the distribution of S_t which is log-normal just like under Π measure, but with mean $S_0 e^{tr}$ and variance $S_0^2 (e^{tr})^2 (e^{\sigma^2 t} - 1)$. Thus, the distribution of S_t under Θ martingale measure is fully determined by time t and noise σ (so-called volatility) and does not depend on growth rate μ .

Since we have found the marginal distribution of S_t under Θ , we can start dealing with the derivatives pricing. We need to know the stock distribution, claim, time of exercising the claim and the riskless interest rate r . The price of the claim X (i.e. derivative price) is then the expected value of the discounted claim under Θ :

$$V_0 = E_{\Theta} \left(B_T^{-1} X \right), \quad (6)$$



where T is time of exercising the claim and B_T^{-1} is discounted zero coupon bond which is defined as $B_0 e^{-rT}$.

3. RISK NEUTRAL DERIVATIVES PRICING

In order to compute the derivative price (6), we have to find the distribution of discounted claim under Θ using the transformation of random variable technique. To be able to do that, it is easier to use normal distribution preferably $N(0,1)$ instead of log-normal distribution of S_t . Distribution of S_t is defined by:

$$S_t = S_0 e^{\left(\left(r - \frac{1}{2} \sigma^2 \right) t + \sigma \sqrt{t} Z \right)}, \quad (7)$$

where Z is $N(0, 1)$ under Θ (Joshi, 2005).

3.1. Pricing a call option

Call option (European) gives the holder the right, but not the obligation, to buy stock at a specified price (strike) at time $t = T$. A rational investor will exercise the option only if the stock value is greater than the strike price k at time T . Thus, the claim X is the function of stock $X = (S_T - k)^+$ at time $t = T$. Primarily, we have to determine the moments of the claim or of the discounted claim under Θ . Using (7) we get density (8) and distribution function (9) of discounted claim $Y = B_T^{-1} X$.

$$f(y) = \frac{\sqrt{\frac{2}{\pi}}}{2\sigma\sqrt{t} e^{\frac{\left(\frac{t\sigma^2}{2} + \log\left(\frac{y+ke^{-rt}}{S_0} \right) \right)^2}{2\sigma^2 t}} (y + ke^{-rt})} \quad (8)$$



$$F(y) = \frac{1}{2} \operatorname{erf} \left(\frac{\sqrt{2} \left(\frac{t\sigma^2}{2} + \log \left(\frac{y + ke^{-rt}}{S_0} \right) \right)}{2\sigma\sqrt{t}} \right) + \frac{1}{2} \quad (9)$$

Having appropriate density function we compute the expected value (6). We have to keep in mind that the claim X and discounted claim Y may reach only positive values. Hence, the price V of the call option is

$$V(S_0, T) = \int_0^{\infty} y f(y) dy, \quad (10)$$

Suppose a stock with constant volatility σ of 20% and constant drift μ of 5% with continuously compounded interest rate r constant at 3%. What is the price of an option to buy the stock for \$8 in 5 years time, given the current stock price of \$5?

The density (8) and the distribution function (9) are displayed in Figure 1. We solve the equation (10) numerically with the outcome of 0.35999. Thus, the price of the call option is \$0.36 at time $t = 0$. All other prices would lead to arbitrary risk-free profits through constructing appropriate replication portfolio of a bond and a stock (Joshi, 2005).

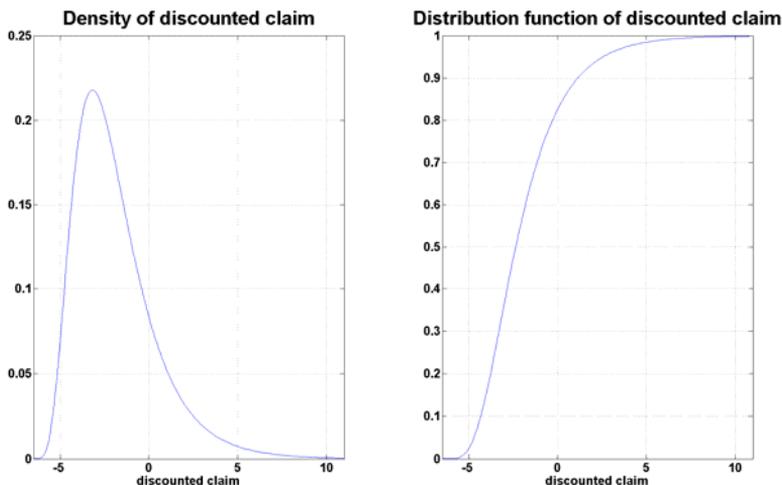
Let us take a look at the Black-Scholes formula (Hull, 2006). Fischer Black and Myron Scholes constructed the self financing replication portfolio of bond and stock in continuous world. Thus, they had to solve a PDE (Partial Differential Equation) to get the correct option price. The PDE has an explicit solution therefore their formula (11) became so popular. Applying formula (11), the call option price equals \$0.36 which is exactly the same price as using risk neutral approach. The explicit solution to B-S for call options is given by

$$V(S_0, T) = S_0 \Phi \left(\frac{\log \frac{S_0}{k} + \left(r + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right) - ke^{-rT} \Phi \left(\frac{\log \frac{S_0}{k} + \left(r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right), \quad (11)$$



where the notation $\Phi(x)$ denotes the distribution function of $N(0,1)$.

Figure 1: Density and distribution function of discounted call option claim with $\sigma = 0.2$, $k = 8$, $S_0 = 5$, $T = 5$, $r = 0.03$.



3.2. Pricing a put option

Put option (European) gives the holder the right, but not the obligation, to sell stock at a specified price (strike) at time $t = T$. A rational investor will exercise the option only if the stock value is less than the strike price at time T . Thus, the claim X is the function of stock $X = (k - S_T)^+$ at time $t = T$. Applying the transformation of random variable technique, we get the density (12) and the distribution function (13) of discounted claim Y .

$$f(y) = \frac{e^{rt} \sqrt{\frac{2}{\pi}}}{2\sigma\sqrt{t} e^{\frac{\left(\log\left(\frac{k - ye^{rt}}{S_0}\right) - t\left(r - \frac{\sigma^2}{2}\right)\right)^2}{2\sigma^2 t}} (k - ye^{rt})} \quad (12)$$



$$F(y) = \frac{1}{2} - \frac{1}{2} \operatorname{erf} \left(\frac{\sqrt{2} \left(\log \left(\frac{k - ye^{rt}}{S_0} \right) - t \left(r - \frac{\sigma^2}{2} \right) \right)}{2\sigma\sqrt{t}} \right).$$

(13)

Having appropriate density function, we compute the expected value (10) where $t=0$ for $y < 0$ and $t=y$ for $y > 0$.

Density and distribution functions of discounted claim under Θ martingale measure for $\sigma = 0.2$, $k = 8$, $S_0 = 5$, $T = 5$, $r = 0.03$ are shown in Figure 2. Again, we solve the integral (10) numerically. The price of the put option is \$2.2457. We get exactly the same price applying the Black-Scholes formula for put option. The Black-Scholes formula for put option is given by explicit PDE solution (4), see (Hull, 2006) for details.

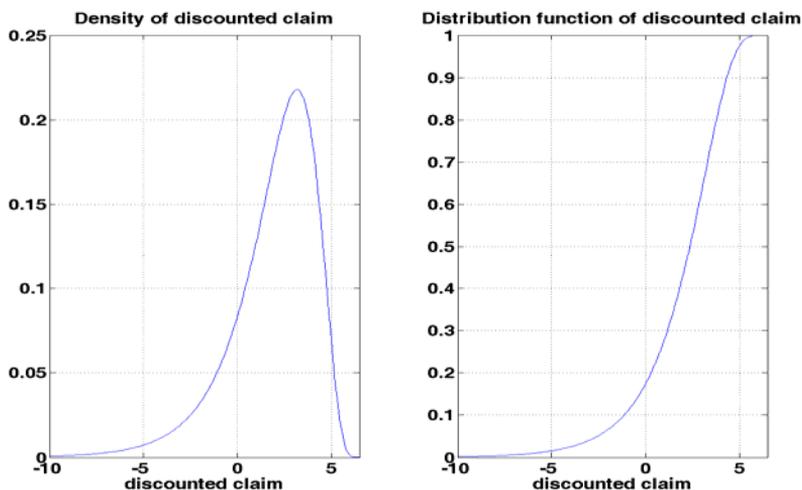
$$V(S_0, T) = ke^{-rT} \Phi \left(\frac{\log \frac{S_0}{k} + \left(r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right) - S_0 \Phi \left(\frac{\log \frac{S_0}{k} + \left(r + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right),$$

(14)

where the notation $\Phi(x)$ denotes the distribution function of $N(0,1)$.



Figure 2: Density and distribution function of discounted put option claim with $\sigma = 0.2$, $k = 8$, $S_0 = 5$, $T = 5$, $r = 0.03$.



3.3. Pricing an arbitrary derivative

Let us suppose a contract paying off some arbitrary function of stock price at time $t = T$. The contract also has guaranteed minimum payout and maximum payout according to gains of the stock. More precisely, it is a five-year contract which pays out 90% of the ratio of the terminal and initial values of the stock. Or it pays out 130% of initial stock price if otherwise it would be less, or 180% of initial stock price if otherwise it would be more. How much is this contract worth at time $t = 0$? Thus, the claim X is

$$X = \min \left\{ \max \left\{ 1.3S_0, 0.9 \frac{S_T}{S_0} \right\}, 1.8S_0 \right\}, \quad (15)$$

To price this contract, we have to compute the expected value of the discounted claim under Θ martingale measure (6) in the same way as pricing call and put options. The best way to do that is separating the claim into three partial claims. Then, we determine the value of each partial claim using a corresponding interval of stock price S at time $t = T$. At the end, we sum all values of partial claims and we get the total value of the derivative (15). The partial claims and corresponding intervals are shown in (16)-(18).



$$S_T \in \left(0, \frac{1.3}{0.9} S_0^2\right) X_1 = 1.3 S_0. \quad (16)$$

$$S_T \in \left(\frac{1.3}{0.9} S_0^2, 2S_0^2\right) X_2 = 0.9 \frac{S_T}{S_0}. \quad (17)$$

$$S_T \in (2S_0^2, \infty) X_3 = 1.8 S_0. \quad (18)$$

We assume stock drift $\mu = 7\%$, stock volatility $\sigma = 15\%$, initial stock value $S_0 = 1$ and interest rate $r = 6.5\%$. We compute the value of the partial claim (16) at first. To do that, we have to find the probability $P\left(S_T \in \left(0, \frac{1.3}{0.9} S_0^2\right)\right)$ under Θ which is the probability of exercising the first claim. The distribution of S_T is given by (7). The density and distribution functions of S_T are shown in Figure 3. The resulting probability is

$$P_{\Theta}\left(S_T \in \left(0, \frac{1.3}{0.9} S_0^2\right)\right) = 0.616.$$

Hence, using the derivative pricing formula (6) multiplied by the corresponding probability, we get the value of first partial claim.

$$V_0^1 = E_{\Theta}^1\left(B_T^{-1} X_1\right) P_{\Theta}\left(S_T \in \left(0, \frac{1.3}{0.9} S_0^2\right)\right) = e^{-rT} X_1 P_{\Theta}\left(S_T \in \left(0, \frac{1.3}{0.9} S_0^2\right)\right).$$

Partial claim X_1 has the value of 0.57863.

Applying the same procedure, we find value of X_3 (18). Having

$$V_0^3 = E_{\Theta}^3\left(B_T^{-1} X_3\right) P_{\Theta}\left(S_T \in (2S_0^2, \infty)\right) = e^{-rT} X_3 P_{\Theta}\left(S_T \in (2S_0^2, \infty)\right)$$

where

$$P_{\Theta}\left(S_T \in (2S_0^2, \infty)\right) = 0.1029.$$

Partial claim X_3 has the value of 0.1338.

Now we are to price the remaining partial claim X_2 . Again we apply (6)

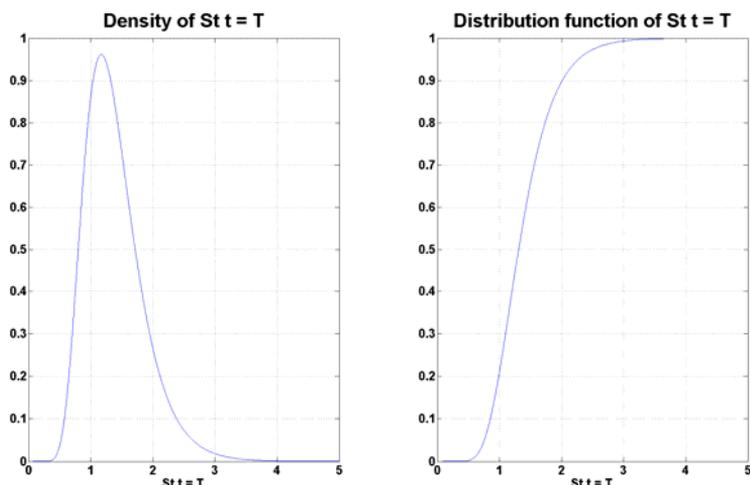


$$V_0^2 = E_{\Theta}^2(B_T^{-1}X_2) = 0.9e^{-rT} \int_{-\infty}^{\infty} t f(y) dy,$$

where $t = y$ for $y \in (\frac{1.3}{0.9}S_0^2, 2S_0^2)$ and $t = 0$ otherwise, $f(y)$ denotes density function of S_T under Θ martingale measure. The value of partial claim X_2 is 0.30588.

The value of the derivative is 1.01831835, which is sum of values of the partial claims. Constructing the replication portfolio to fully hedge the claim (15) and solving corresponding PDE, we get the value of 1.01831835 which is exactly the same price as applying the risk-neutral approach (Baxter, 1999).

Figure 3: Density and distribution function of S_T with $\sigma = 0.15$, $S_0 = 1$, $T = 5$, $r = 0.065$.



4. MARKET PRACTICE

Let me take a look at real market options prices. How close are the market prices of options to those predicted by Black-Scholes and risk-neutral approach using lognormal distribution? Are the probability distributions of the asset prices really lognormal at any future time? Traders use the Black-Scholes model, but not exactly in the way that Black and Scholes originally intended. The risk-neutral approach is general enough to produce correct prices under changed conditions in the real market. A lot of information about underlying, including probability distribution, can be derived from options prices. I will analyze the distribution of CEZ stock price using CEZ warrants prices.



4.1. Data description

I chose the options on CEZ stock due to their relatively high market activity comparing to the options on the other Czech stocks. The only optional instruments on CEZ stock are warrants denominated in EUR and traded on Boerse Stuttgart. I used all priced call warrants on CEZ traded on Boerse Stuttgart as of 23 April 2009. See Appendix B for the complete list of used warrants. Market mid prices were taken from Boerse Stuttgart via Bloomberg. Warrants are very similar to call options, and will often confer the same rights as an equity option and can even be traded on secondary markets. However, they have several key differences. Firstly, warrants are issued by private parties, typically the corporation on which a warrant is based and by investment banks, rather than a public options exchange. Secondly, warrants are not standardized like exchange-listed options. While investors can write stock options on the CEZ share, they are not permitted to do so with CEZ-listed warrants. As all CEZ warrants traded on Boerse Stuttgart are written by banks, they are not dilutive¹, hence we can handle the warrants as options, respecting the ratio naturally. The ratio determines how many warrants are needed in order to acquire the right to buy or sell one unit of the underlying instrument. Because all considered warrants are American-style calls on dividend paying stock, we are forced to modify our pricing formula for European call (10). Unlike the European exercising style, American style means that the right to exercise the warrant can be invoked on any trading day during the life of the warrant. It can be shown that it is never optimal to exercise an American call option on non-dividend paying stock before the expiration date. When underlying stock pays off dividends, it is optimal to exercise the option either at a time immediately before the stock goes ex-dividend or at a time of expiration (Hull, 2006). CEZ pays off the dividend once a year, more importantly, once during the life of all considered options. Using Black's approximation, we price two Europeans options with exercise dates at maturity and at time immediately before dividend. The value of American-style option is then the greater of the values (Hull, 2006).

4.2. Volatility smile

Volatility is an unobservable parameter and must be estimated. If there are no tradable options on the market we can estimate the volatility from a history of the stock price. We face the problem of the length of time series in this case. For tradable options, we determine the volatility from the option's market price and the pricing formula. Hence, this kind of volatility is implied. Our pricing formulas assume that the probability density function (PDF) of the underlying asset is lognormal

¹Warrants issued by the company itself are dilutive. When the warrant issued by the company is exercised, the company issues new shares of stock, hence the number of outstanding shares increases.



with constant volatility at any given future time. This assumption is not the one made by traders. According to empirical studies, the market assumes the probability distribution of an equity price with heavier left tail and less heavy right tail than the lognormal distribution. The reason is that traders think that the probability of a larger downward movement in the stock price is higher than that predicted by the lognormal probability distribution. For standard options, traders use volatility smiles to allow for nonlognormality. They adjust the volatility parameter according to the volatility smile and then they use lognormal distribution. The volatility smile defines the relationship between the implied volatility of an option and its strike price. For equity options, the volatility smile tends to be downward sloping. This fact means that call options with strike price below the market price of the stock (in-the-money) tend to have higher implied volatility than call options with strike price above the market price of the stock (out-of-the-money). Hence, in-the-money call options are relatively more expensive than out-of-the-money calls. We can also define the implied volatility as a function of strike and option's time to maturity which is standard practice in the market. We call this relationship the volatility surface.

Having an option market price, we need the pricing formula to derive the implied volatility of particular option. Carr has further developed and generalized the Geske-Johnson approach and has introduced general approach to the valuation of the American-style option, see appendix and (Carr, 1995) for details. Since CEZ pays off dividend once during the life of all considered options, it is optimal to exercise the option either immediately before the stock goes ex-dividend or at the option's maturity. For that reason, we consider $n=2$ for $\omega_1(\delta_k)$ and for $\omega_2(\delta_d)$ in (28) and in (29) respectively which leads to using the bivariate normal distribution in the price formula. Using the Carr risk-neutral approach, the value of American call is

$$V_0 = k\omega_1(\delta_k) - S_0\omega_2(\delta_d), \quad (19)$$

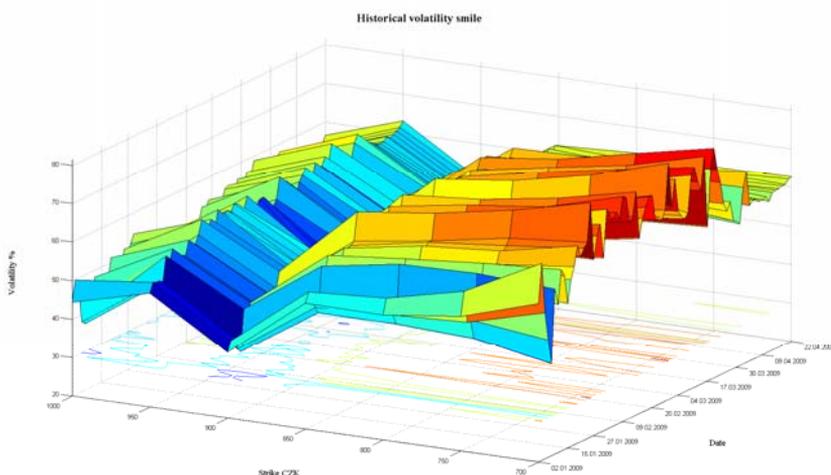
where $\omega_1(\delta_k)$ and $\omega_2(\delta_d)$ are defined in appendix.

Solving the equation (19) for σ , we obtain the option's implied volatility. I solved the equation for all options with any market activity maturing on 17 June 2009 for all trading days since 1 January 2009 till 23 April 2009. The resulting implied volatility smile time series is displayed in Figure 4. We can observe that the volatility tends to increase with the option's increasing time to maturity and tends to decrease with increasing strike price. This outcome is in accordance with the theory. Traders often call this downward sloping shape as a volatility skew. The warrant with strike value of CZK 900 seems to have constantly lower implied volatility, hence it seems to be constantly undervalued. To explain this discrepancy between the theory and the



market, we have to take a look at market activity. The trading volume is far higher than by the other warrants and hence the bid-offer spread is much smaller for all analyzed days. Due to higher liquidity on this warrant, its price represents the market opinion better than the prices of the other warrants. I will use weights on the warrants subject to their liquidity to express market price significance later in the paper.

Figure 4: Volatility skew for American call options on CEZ stock



4.3. Implied distribution

The variance that is implied by an option's price is the market ex ante estimate of the underlying asset's return volatility over the remaining life of an option. More interestingly, it is possible to derive the higher moments of future asset values from the market options' prices. These can be extracted in the form of an ex ante risk-neutral probability distribution of the underlying asset at particular future date. Rather than specifying the underlying asset price dynamics to infer the risk-neutral density function, it is possible to make an assumption about the functional form of density function itself and to recover its parameters by minimizing the distance between the observed prices and those that are generated by the assumed probability distribution.

Since all considered warrants on CEZ stock are American-style calls with one dividend payout during warrants life, it is important to know whether it is optimal to exercise the warrant immediately before ex-dividend date or at maturity. To identify this, I used Black's approximation approach. Black's approximation takes account of early exercise in call options. This involves calculating the price of European options that mature at option's maturity date T and at time immediately before the stock goes ex-dividend (t_1), and then setting the American price equal to the greater of the two. We start



considering the possibility of early exercise just prior to the ex-dividend date (i.e., at time t_1). If the warrant is exercised at time t_1 , the investor receives

$$S_{t_1} - K.$$

If the warrant is not exercised, the stock price drops to $S_{t_1} - D_1$, where D_1 is the dividend. The dividend on CEZ is determined in amount of CZK 50 per stock for 2009. The ex-dividend date is 13 May 2009. Because the lower bound for call options is $S_{t_1} - D_1 - ke^{-r(T-t_1)}$ (Hull, 2006), it follows that if

$$S_{t_1} - D_1 - ke^{-r(T-t_1)} \geq S_{t_1} - D_1,$$

that is,

$$D_1 \leq k(1 - e^{-r(T-t_1)})$$

it cannot be optimal to exercise the option at time t_1 . On the other hand, if

$$D_1 > k(1 - e^{-r(T-t_1)}) \quad (20)$$

for any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise the option at time t_1 . The inequality in (20) will tend to be satisfied when ex-dividend date t_1 is fairly close to the maturity of the option T and the dividend is large. I used forward mid rate between PRIBOR and PRIBID as the risk-free interest rate. Applying this method to all considered warrants, I got the information whether it is optimal to exercise the warrant at time t_1 or at time T . The outcomes are displayed in Figure (6). Using this information, we can derive the implied distribution of CEZ equity price.

The price of the European call option is given by (6) at time t where the claim is $X = (S_T - k)^+$. Unlike the density function used in chapter "Pricing a call option", I assume that the risk-neutral distribution of stock is the mixture of two lognormals rather than lognormal at any future time. We can use the arbitrary density function which fulfills the criteria of probability density function and has finite variance. The problem with using other than the Gaussian density functions is that the underlying price distribution changes as the holding period changes. Under the assumption of lognormally distributed daily returns, the arbitrary length holding period price distributions must also



be lognormal. No other finite variance distribution is similarly stable (Bahra, 1997). Moreover, the functional form assumed for the density function should be relatively flexible. In particular, it should be able to capture the main contributions to the volatility smile, namely the skewness and the kurtosis of the underlying distributions. A weighted sum of independent lognormal density functions fits these criteria. The probability distribution of any future stock price is given by

$$f(S_T) = \theta L(S_T | \mu_1, \sigma_1, S_0) + (1 - \theta) L(S_T | \mu_2, \sigma_2, S_0), \quad (21)$$

where $L(S_T)$ is the lognormal density function given by (22) and θ is the probability weight satisfying the condition that $\theta \in [0, 1]$.

$$L(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi\tau}} \exp\left(-\frac{(\log S_T - \log S_0 - (\mu - 0.5\sigma^2)\tau)^2}{2\sigma^2\tau}\right), \quad (22)$$

where τ denotes time to future date on which we derive the density function and $\{\mu_1, \sigma_1, \mu_2, \sigma_2, \theta\}$ are the unknown parameters.

Transforming the stock price density function ((22)) for $X = (S_T - k)^+$, we get the density function of call warrant price given by

$$f(x) = \frac{1}{(x+k)\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{\left(\log \frac{x+k}{S_0} - (\mu - 0.5\sigma^2)\tau\right)^2}{2\sigma^2\tau}\right). \quad (23)$$

Thus the fitted value for a call price, given parameters $\{\mu_1, \sigma_1, \mu_2, \sigma_2, \theta\}$, is given by

$$\hat{C}_t(k, \tau | \mu_1, \sigma_1, \mu_2, \sigma_2, \theta) = e^{-r\tau} \left(\theta \int_0^\infty x f(x | \mu_1, \sigma_1) dx + (1 - \theta) \int_0^\infty x f(x | \mu_2, \sigma_2) dx \right). \quad (24)$$

Having the fitted values of calls warrants, we can minimize the sum of squared errors, with respect to the five distributional parameters and risk-free rate r , between the warrant prices generated by the mixture distribution model and those actually observed in the market. In practice, I used the corresponding mid rate between PRIBID and PRIBOR to fix r , and thereby reduced the dimensionality of the problem. We should include put warrants in our minimization problem as well, since they are priced off the same underlying distribution.



There were only two put warrants traded on Boerse Stuttgart at the time of writing. Both of them are traded close to at-the-money² state. In my opinion, including these warrants in the optimization problem will not provide significantly better information about implied distribution. Moreover, both puts are American-style warrants on dividend paying stock during their life. This means that it can be optimal to exercise them at any time during their life. This fact would lead to computing problems (Melick et al., 1994). To avoid the arbitrage opportunities, the mean of the implied density function should equal the forward price of underlying asset. The forward price of the underlying assets is equal to $e^{r\tau}S_0$ in risk-neutral world. The expected value of implied distribution (21) is given by

$$E(S_T) = \theta L \int_0^\infty S_T L(S_T | \mu_1, \sigma_1, S_0) dS_T + (1 - \theta) L \int_0^\infty S_T L(S_T | \mu_2, \sigma_2, S_0) dS_T. \quad (25)$$

Some warrants are fairly illiquid, particularly at deep in-the-money and out-of-the-money strike prices. Warrant prices at these outer strikes are less informative about market expectations, or are not available. Hence, I decided to use weights on each warrant which equal trading volumes on each warrant on 24 April 2009. I excluded non-priced warrants from the estimation sample. Given observations of call prices c_i , I estimated the parameters, $\{\mu_1, \sigma_1, \mu_2, \sigma_2, \theta\}$, of the implied two-lognormal density function minimizing the objective function

$$\min_{\{\mu_1, \sigma_1, \mu_2, \sigma_2, \theta\}} \sum_{i=1}^{N_c} w_i \left(c_{i_i} - \hat{C}_{i_i}(k, \tau | \mu_1, \sigma_1, \mu_2, \sigma_2, \theta) \right)^2 + \left(E(S_T) - e^{r\tau} S_0 \right)^2, \quad (26)$$

subject to

$$\sum_{i=1}^{N_c} w_i = 1 \text{ and } w_i \geq 0 \forall i,$$

where N_c is number of call warrants in the estimation sample. Minimizing objective function (26), we have to ensure that the integral of the resulting implied density function (21) from 0 to ∞ is equal to one, i.e. $\int_0^\infty f(S_T) dS_T = 1$. Furthermore, the constraints on parameters are $\sigma_1, \sigma_2 > 0$ and $\theta \in (0, 1)$. The parameters' estimates are displayed in Table 1. Figure 5 shows the two-lognormal mixture implied density

²The market price of the underlying asset is near the strike price



function of CEZ stock with its weighted component lognormal density functions on 17 June 2009. Additionally, Figure 5 shows the lognormal density function with the same mean and standard deviation as the implied distribution. Figure 5 displays the expected value of implied distribution and forward price as well. We can observe that the expected value of CEZ stock derived from implied distribution and forward price of CEZ stock are practically identical which was one of the criteria used by minimizing the objective function. We can observe that the resulting implied distribution of CEZ stock price has greater kurtosis than lognormal distribution and gives higher price to in-the-money warrants and lower price to out-of-the-money warrants than the initially considered lognormal distribution. Figure 6 shows the difference between warrants' market prices, i.e. observed prices, and their prices implied by two-lognormal mixture distribution. The average difference is 0.28 EUR. We can observe less accurate fit by less liquid warrants and vice versa. Each warrant is labeled by its optimal exercise time. 1 denotes exercising the warrant at time immediately before the stock goes ex-dividend and 0 denotes the time of warrant maturity. We can observe that early exercise does not affect the quality of the fit. Unfortunately, all warrants on CEZ are only traded at discretely spaced strike price levels and for very limited range either side of the at-the-money strike (ATM), which is from CZK 700 to CZK 1550 in our case. ATM price is CZK 835. Hence, all implied density functions interpolate between observed strike prices and extrapolate outside of their range to model the tail probabilities. The tails, especially the left one in the call out-of-the money area, are given by the function formula rather than by observations. This issue concerns to almost all warrants and option on the market, even those traded on developed, highly liquid markets.

Table 1: Parameters estimates

$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\theta}$
-0.684	0.165	0.555	0.369	0.22

Implied distribution is used primarily for pricing non traded derivatives on underlying asset, but implied distribution contains much more information. It provides us with information about future behaviour of CEZ equity in terms of probabilities. For example, the mean of 787 CZK is the expected future value of CEZ stock on 17 June 2009. The mode of CZK 770, on the other hand, is the most likely future outcome on this date. The standard deviation of 121 CZK is the measure of the uncertainty around the mean. The positive skewness of 0.2 tells us that there is less probability attached to outcomes higher than the mean than to outcomes below the mean. The kurtosis of 3.48 is a measure of how peaked the distribution is and the likelihood of extreme outcomes: the greater the kurtosis, the fatter the tails of the distribution. These



statistics provide a useful way of tracking the behaviour of implied density functions over the life of a single contract and making comparisons across contracts. Implied distribution may also be useful for analyzing market reactions to company earnings releases, money market operations, government bond auctions, etc. Implied summary statistics for CEZ stock on 17 June 2009 are displayed in Table 2. The higher moments can be derived directly from density function using n th moments of implied density function. The n th moment μ'_n is given by

$$\mu'_n(S_T) = \int_0^{\infty} S_T^n f(S_T) dS_T,$$

where $f(S_T)$ is two-lognormal mixture density function (21). For $n = 1$, we get expected value of $f(S_T)$. Further, we need to compute n th central moments μ_n using formula:

$$\mu_n(S_T) = E\left\{(S_T - E(S_T))^n\right\}$$

For $n = 2$ we get the variance of S_T . Skewness $\mu_3(U)$ is normalized third central moment given by

$$\mu_3(U) = \frac{\mu_3(S_T)}{\mu_2(S_T)\sqrt{\mu_2(S_T)}},$$

and kurtosis $\mu_4(U)$ is the normalized fourth central moment given by

$$\mu_4(U) = \frac{\mu_4(S_T)}{(\mu_2(S_T))^2}.$$

Table 2: Implied risk-neutral summary statistics for CEZ stock on 17 June

mean	787
standard deviation	121
skewness	0.2
kurtosis	3.48

It is important to remember that the derived implied density function is risk-neutral, that is, it is equivalent to the true market density function only when the investors are risk neutral. Hence, the implied summary statistics are risk-neutral as well. In reality the investors are likely to be risk averse, and warrant prices will incorporate these preferences towards risk as well as beliefs about future outcomes. Distinguishing between these two factors would require specification of the



unobservable aggregate market utility function and estimation of the corresponding coefficient of risk aversion. Rubinstein assumes a market risk premia of between 3.3 and 5 per cent for an equity index and blue chips equities. More importantly, he finds that the subjective distribution is only slightly shifted to the right relative to the risk-neutral distribution, and the qualitative shapes of the two distributions are quite similar (Rubinstein, 1994).

Figure 5: Two-lognormal mixture implied density function of CEZ stock on 17 June 2009 with its weighted component lognormal density functions, lognormal density function with the same mean and standard deviation as the implied distribution, expected value of implied distribution, forward price of CEZ stock

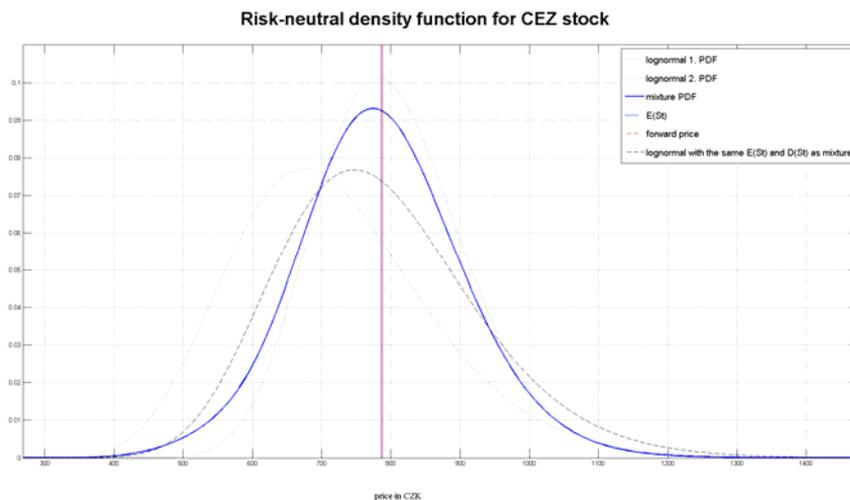
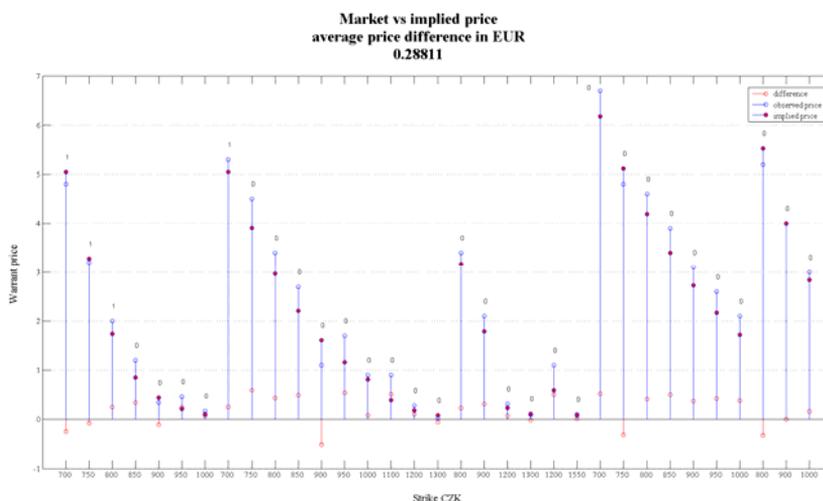




Figure 6: Absolute differences between warrants market prices, i.e. observed prices, and their prices implied by two-lognormal mixture distribution. Optimal exercise time where 1 denotes the time immediately before the stock goes ex-dividend and 0 maturity.



5. CONCLUSIONS

In this paper, I have presented the risk-neutral option pricing approach and have shown its application in pricing real market derivatives. I have derived the probability distribution of an underlying asset price, stock in particular, from the binomial tree process of the stock price in real world. Since real world is irrelevant for pricing, I changed the measure and have derived risk-neutral distribution of the stock price. The discrete trees are only an approximation of the way that prices actually move. In practice, a price can change at any instant, rather than just at some fixed tick-times. Using central limit theorem, I came over to continuous time. Under assumption of constant volatility is the distribution of stock price lognormal. I have priced three different derivatives and shown no bias between prices produced by risk-neutral approach and those produced by replication strategy approach. The replication strategy approach leads to solving partial differential equations PDE (heavily used Black-Scholes formula for instance). The presented risk-neutral approach is suitable for pricing any European-style claim function of underlying asset. I see its main advantage in its relative simplicity, compared to the continuous stochastic process approach which leads to solving the stochastic differential equations SDE.

The results in this paper provide strong evidence of nonlognormality of CEZ stock price. I used American-style call warrants on CEZ with



dividend pay off during the life of warrant to construct the volatility smile time series. As CEZ pays off the dividend, we have to take into account the possibility of early exercise. We can observe that volatility is the function of strike price rather than a constant value. Furthermore, I have derived the implied distribution of CEZ stock price from warrants prices observed on the market. I used two-lognormal mixture density function which is, in my opinion, flexible enough to capture the main contributions to the volatility smile, namely the skewness and the kurtosis of the underlying distribution. Minimizing the objective function subject to highly nonlinear constraint, I have found parameters estimates of the implied density function. Not all warrants are equally liquid and thus their prices are not equally reliable. Hence, I used traded volume on each warrant as weights.

The resulting implied distribution of CEZ stock price has greater kurtosis than the lognormal distribution and gives higher price to in-the-money warrants and lower price to out-of-the-money warrants than initially considered lognormal distribution. Using implied density function, we can price non-traded derivatives or search for mispriced ones. Implied distribution provides us with information about the future in behaviour of underlying asset in terms of probabilities as well. I have derived higher moments of CEZ stock price like skewness and kurtosis. These statistics provide a useful way of tracking the behaviour of implied density functions over the life of a single contract and making comparisons across contracts. Observing day-to-day changes in these statistics, we may also analyze market reactions and future expectations to company earnings releases, money market operations, government bond auctions, etc.



APPENDIX

Appendix A

General valuation formula to the timing option

General valuation formula to the timing option³ is defined by

$$V_0 = k\omega_1(\delta_k) - S_0\omega_2(\delta_{S_0}) \quad (27)$$

where

$$\begin{aligned} \omega_1(\delta_k) = & e^{-\delta_k \Delta t} N_1 \left(d_1 \left(\frac{Pe^{-\delta \Delta t}}{P_1^*} \right) \right) + \\ & + e^{-\delta_k 2\Delta t} N_2 \left(-d_1 \left(\frac{Pe^{-\delta \Delta t}}{P_1^*} \right), d_1 \left(\frac{Pe^{-\delta 2\Delta t}}{P_2^*} \right); -\sqrt{\frac{t_1}{T}} \right) + \dots + \\ & + e^{-\delta_k \tau \Delta t} N_n \left(-d_1 \left(\frac{Pe^{-\delta \Delta t}}{P_1^*} \right), \dots, -d_1 \left(\frac{Pe^{-\delta(n-1)\Delta t}}{P_{n-1}^*} \right), d_1(Pe^{-\delta \tau}), \Omega_n \right), \end{aligned} \quad (28)$$

and

$$\begin{aligned} \omega_2(\delta_d) = & e^{-\delta_d \Delta t} N_1 \left(d_2 \left(\frac{Pe^{-\delta \Delta t}}{P_1^*} \right) \right) + \\ & + e^{-\delta_d 2\Delta t} N_2 \left(-d_2 \left(\frac{Pe^{-\delta \Delta t}}{P_1^*} \right), d_2 \left(\frac{Pe^{-\delta 2\Delta t}}{P_2^*} \right); -\sqrt{\frac{t_1}{T}} \right) + \dots + \\ & + e^{-\delta_d \tau \Delta t} N_n \left(-d_2 \left(\frac{Pe^{-\delta \Delta t}}{P_1^*} \right), \dots, -d_2 \left(\frac{Pe^{-\delta(n-1)\Delta t}}{P_{n-1}^*} \right), d_2(Pe^{-\delta \tau}), \Omega_n \right), \end{aligned} \quad (29)$$

where $P = S_0/k$, $\Delta t = \frac{\tau}{n}$ (where τ is option's time to maturity $\tau = T - t$), δ_k and δ_d are the corresponding dividend yields (for American call on stock $\delta_d = r$, for American put on stock $\delta_k = r$). Further, for American call $\delta = r - \delta_k$, for American put $\delta = r - \delta_d$.

³Timing option gives the buyer the right to a choice of different exercise dates.



Further, N_i is the standard i -variate distribution function with correlation matrix Ω_n where

$$d_1(G) = \frac{\ln(G) + \frac{\sigma^2 \tau}{2}}{\sqrt{\sigma^2 \tau}}, \quad (30)$$

and

$$d_2(G) = d_1(G) - \sqrt{\sigma^2 \tau}, \quad (31)$$

where σ is the volatility. Further, P_k^* is the unique value of the price ratio P . Price ratio P is the critical value of inequality which tells us that the option will not be exercised at mid-life if the opportunity cost of exercise exceeds the cash proceeds of exercise (Carr, 1995). For instance, P_1^* is the solution to

$$\begin{aligned} & P_1^* \left[e^{-\delta_k \Delta t} N_1 \left(d_1 \left(\frac{P_1^* e^{-\delta \Delta t}}{P^*} \right) \right) + e^{-\delta_k 2 \Delta t} N_2 \left(-d_1 \left(\frac{P_1^* e^{-\delta \Delta t}}{P^*} \right), d_1 \left(P_1^* e^{-\delta 2 \Delta t} \right); -\sqrt{\frac{t_1}{T}} \right) \right] - \\ & - \left[e^{-\delta_d \Delta t} N_1 \left(d_2 \left(\frac{P_1^* e^{-\delta \Delta t}}{P^*} \right) \right) + e^{-\delta_d 2 \Delta t} N_2 \left(-d_2 \left(\frac{P_1^* e^{-\delta \Delta t}}{P^*} \right), d_2 \left(P_1^* e^{-\delta 2 \Delta t} \right); -\sqrt{\frac{t_1}{T}} \right) \right] \\ & = P_1^* - 1, \end{aligned} \quad (32)$$

where P^* is the solution to

$$P^* e^{-\delta_k \Delta t} N_1 \left(d_1 \left(P^* e^{-\delta \Delta t} \right) \right) - e^{-\delta_d \Delta t} N_1 \left(d_2 \left(P^* e^{-\delta \Delta t} \right) \right) = P^* - 1. \quad (33)$$

See (Carr, 1995) for details



Appendix B

Table 3: Call warrants on CEZ stock used to derive two-lognormal mixture implied density function of CEZ. These warrants are traded on Boerse Stuttgart as at 23 April 2009. Warrant codes were taken from Bloomberg .

<i>warrant code</i>	<i>maturity date</i>	<i>strike in CZK</i>
CM5LQM GR Equity	6/17/2009	900
CM5LQN GR Equity	6/17/2009	1000
CM6ATV GR Equity	6/17/2009	700
CM6ATW GR Equity	6/17/2009	750
CM6ATX GR Equity	6/17/2009	800
CM6ATY GR Equity	6/17/2009	850
CM6ATZ GR Equity	6/17/2009	950
CM0NXJ GR Equity	9/16/2009	1200
CM0NXK GR Equity	9/16/2009	1300
CM5LQQ GR Equity	9/16/2009	900
CM5LQR GR Equity	9/16/2009	1000
CM5LQS GR Equity	9/16/2009	1100
CM6AUA GR Equity	9/16/2009	700
CM6AUB GR Equity	9/16/2009	750
CM6AUC GR Equity	9/16/2009	800



Equity			
CM6AUD GR Equity		9/16/2009	850
CM6AUE GR Equity		9/16/2009	950
EB5EKE GR Equity		9/30/2009	800
EB5D93 GR Equity		9/30/2009	1200
EB5D94 GR Equity		9/30/2009	1300
EB5EKF GR Equity		9/30/2009	900
DB4L87 GR Equity		12/10/2009	1200
DB4L88 GR Equity		12/10/2009	1550
CM6AUF GR Equity		12/16/2009	700
CM6AUG GR Equity		12/16/2009	750
CM6AUH GR Equity		12/16/2009	800
CM6AUJ GR Equity		12/16/2009	850
CM6AUK GR Equity		12/16/2009	900
CM6AUL GR Equity		12/16/2009	950
CM6AUM GR Equity		12/16/2009	1000
EB5ET2 GR Equity		3/31/2010	800
EB5ET3 GR Equity		3/31/2010	900
EB5ET4 GR Equity		3/31/2010	1000



6. REFERENCES

- [1] Ait-Sahalia Y and Lo A W (1998): Nonparametric estimation of state-price densities implied in financial asset prices. *Journal of Finance*, 53(2): 499-547
- [2] Bahra B (1997): Implied risk-neutral probability density functions from options prices. *Bank of England working Paper*, (66)
- [3] Baxter M and Rennie A (1999) *Financial Calculus*. Cambridge University Press, Cambridge.
- [4] Bliss R R and Panigirtzoglou N (2002): Testing the stability of implied probability density functions. *Journal of Banking and Finance*, 26(3)
- [5] Carr P (1995) The Valuation of American Exchange Options with Application to Real Options. In Trigeorgis L: *Real Options in Capital Investment: New Contributions*. Praeger Publishers, Westport CT, 109-120.
- [6] Glatzer E and Scheicher (2003): Modelling the Implied Probability of Stock Market Movements. *ECB Working Paper* (212)
- [7] Hoerdahl P (1999): Estimating the implied distribution of the future short-term interest rate using the Longstaff-Schwartz model. *ECB Working Paper*
- [8] Hull J C (2006) *Options, Futures, and Other Derivatives*. 6, Prentice Hall,
- [9] Joshi M (2005) *The Concepts and Practice of Mathematical Finance*. Cambridge University Press, Cambridge
- [10] Mandan D B and Milne F (1994): Contingent claims valued and hedged by pricing and investing in a basis. *Mathematical Finance*, 4, 223-245
- [11] Melick W R and Thomas C P (1994): Recovering an Asset's Implied PDF from Options Prices: An Application to Crude Oil during the Gulf Crisis. *Federal Reserve Board working paper*
- [12] Rubinstein M (1994): Implied Binomial Trees. *Journal of Finance*, 49 (3), 771-818
- [13] Shimko D C (1993): Bounds of probability. *Risk*, 6(4), 33-37